The Development of Algebraic Thinking
A Vygotskian Perspective

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Abstract: Vygotsky asserted that the student who had mastered algebra had attained "a new higher plane of thought", a level of abstraction and generalization which transformed the meaning of the lower (arithmetic) level. He also affirmed the importance of the mastery of scientific concepts for the development of the ability to think theoretically, and emphasized the mediating role of semiotic forms and symbol systems in developing this ability. Although historically in mathematics and traditionally in education, algebra followed arithmetic, Vygotskian theory supports the reversal of this sequence in the service of orienting children to the most abstract and general level of understanding initially. This organization of learning activity for the development of algebraic thinking is very different from the introduction of elements of algebra into the study of arithmetic in the early grades. The intended theoretical (algebraic) understanding is attained through appropriation of psychological tools, in the form of specially designed schematics, whose mastery is not merely incidental to but the explicit focus of instruction. The author's research in implementing Davydov's Vygotskian-based elementary mathematics curriculum in the U.S. suggests that these characteristics function synergistically to develop algebraic understanding and computational competence as well.

Kurzreferat: Vygotsky ging davon aus, dass Lernende, denen es gelingt, Algebra zu beherrschen, „ein höheres gedankliches Niveau" erreicht hätten, eine Ebene von Abstraktion und Generalisierung, welche die Bedeutung der niederen (arithmetischen) Ebene verändert. Er bestätigte auch die Relevanz der Beherrschung von wissenschaftlichen Begriffen für die Entwicklung der Fähigkeit, theoretisch zu denken und betonte dabei die vermittelnde Rolle von semiotischen Formen und Symbolsystemen für die Ausformung dieser Fähigkeit. Obwohl mathematik-historisch und traditionell erziehungswissenschaftlich betrachtet, Algebra der Arithmetik folgte, stützt Vygotski's Theorie die Umkehrung dieser Sequenz bei dem Bemühen, Kinder an das abstrakteste und allgemeinste Niveau des ersten Verstehens heranzuführen. Diese Organisation von Lernaktivitäten für die Ausbildung algebraischen Denkens unterscheidet sich erheblich von der Einführung von Algebra-Elementen in das Lernen von Arithmetik während der ersten Schuljahre. Das beabsichtigte theoretische (algebraische) Verstehen wird erreicht durch die Aneignung psychologischer Mittel, und zwar in Form von dafür speziell entwickelten Schemata, deren Beherrschung nicht nur beiläufig erfolgt, sondern Schwerpunkt des Unterrichts ist. Die im Beitrag beschriebenen Forschungen zur Implementierung von Davydov's elementarmathematischen Curriculum in den Vereinigten Staaten, das auf Vygotsky basiert, legt die Vermutung nahe, dass diese Charakteristika bei der Entwicklung von algebraischem Verstehen und von Rechenkompetenzen synergetisch funktionieren.

ZDM-Classification: C30, D30, H20

I. Historical Context

Russian psychologist Lev Vygotsky stated clearly his perspective on algebraic thinking. Commenting on its development within the structure of the Russian curriculum in the early decades of the twentieth century, he wrote:

> "The rise from preconcepts (which the schoolchild's concepts of arithmetic usually are) to true concepts, such as the algebraic concepts of adolescents, is achieved by generalizing the generalizations of the earlier level. At the earlier stage certain aspects of objects had been abstracted and generalized into ideas of numbers. Algebraic concepts represent abstractions and generalizations of certain aspects of numbers, not objects, and thus signify a new departure—a new, higher plane of thought. The new, higher concepts, in turn, transform the meaning of the lower. The adolescent who has mastered algebraic concepts has gained a vantage point from which he sees concepts of arithmetic in a broader perspective". (Vygotsky 1986, p. 202)

Three essential characteristics of the Vygotskian approach to the development of algebraic thinking are explored in this article: initial development from the most generalized conceptual base, ascent from the abstract to the concrete, and appropriation of psychological tools. These are incorporated into the elementary mathematics curriculum researched and developed by Russian psychologist V.V. Davydov and his colleagues. As will be clear during the ensuing discussion, attempts to focus on one of these characteristics (through designations such as subheadings, for example), will not succeed in segregating it from the others, since they are inseparable in Davydov's curriculum. Consequently, all three are in evidence in every example cited.

In order to achieve the higher plane of thought envisioned by Vygotsky, Davydov saw clearly the disadvantages of a curriculum in which the order of the development of concepts was from arithmetic to algebra, as it was early in the twentieth century when Vygotsky made his observations. An elementary curriculum comprised of empirical concepts (or preconcepts as Vygotsky called them in the above quote), was followed by a secondary curriculum where students finally gained access to the realm of theoretical thought in mathematics. Vygotsky held that only theoretical (also designated "scientific") concepts were real concepts; empirical concepts were not true concepts and hence, were designated as preconcepts above.

Empirical concepts are spontaneously derived from everyday experience, often by comparing and contrasting the empirical features of objects or phenomena. This occurs frequently in school settings as well, as for example, when polygons are compared and classified according to the number of their sides. A theoretical understanding of polygons, on the contrary, would be oriented to the central role of the triangle in their genesis, an observation made as far back in history as Aristotle. It is a matter of more than passing significance that mathematics concepts are quintessentially theoretical (or scientific) in nature (cf. Schmittau 1993).

Other scholars who had either influenced (as did the philosopher Hegel) or studied with Vygotsky, also emphasized the role of theoretical thinking early in the child's development. Hegel held that a child should not be kept for too long in an empirical mode of thinking. And D. B. Elkonin, found that the years best suited for theoretical learning were, in fact, the elementary school years, before interest shifted to peer relations and a focus on future careers in adolescence (Elkonin 1975).

Davydov and his colleagues sought therefore, to introduce theoretical or algebraic thinking earlier in the school experience. Consequently, their approach to the reform of the Russian mathematics curriculum in the late 1950s and early 1960s, was undertaken from a quite different theoretical stance than that underlying the reform during the same time period in the United States. While the US was adopting the formalism of the "new mathematics", Davydov (1975, p. 74-75) noted the weakness of the set theoretic approach that formed its foundation, and demonstrated that the laws of operations of set union and intersection did not hold universally. He did this by considering a situation in which a group of wolves, wolf pups, and rabbits, were to be gathered together. If the rabbits were first paired with the pups, and then the wolves were added, all would be well, since the care of the pups might be assumed to take up the wolves' attention, and cause them to ignore the rabbits. If, however, the wolves and the rabbits were paired first, and then the pups were added to this group, it is probable that the wolves would have devoured the rabbits, and thus the associative property would not hold, i.e., $W \cup (R \cup P) \neq (W \cup R) \cup P$ (Davydov 1975, p.74-75). The fact that set theory was clearly not as fully general as it appeared, in that there were limitations on the real world entities that might qualify as "sets" if the important properties of set union and intersection were to apply, constituted a major drawback to building mathematics on sets as a foundation.

Davydov subscribed to the position taken by Bourbaki, who also rejected the set theoretic foundation, asserting that it is not sets, but rather mathematical *structures* that constitute the essential content of mathematics (Bourbaki 1963, p. 251; cited in Davydov 1975, p. 71). Although historically in mathematics and traditionally in education, algebra followed arithmetic, Vygotskian theory with its emphasis on scientific concepts and theoretical understanding, supports the reversal of this sequence in the service of orienting children to the most abstract and general level of understanding from the beginning of their formal schooling. However, given that elementary school children do not possess the sophisticated understandings of mathematicians or even the numerical background of secondary students, it was by no means obvious how instruction might be designed to render algebraic structure preeminent at the elementary level, without imposing a conceptually sterile and largely unlearnable formalism.

It may appear odd, moreover, that those intent upon the pedagogical implementation of Vygotsky's theory, which is essentially a cultural historical theory, would entertain the possibility of reversing the historical order of development, viz., that of arithmetic followed by algebra. Davydov distinguished, however, between what he termed "external chronologism" (Davydov 1975, p.97) and the internal logic of the development of a subject, in this case, mathematics. He also called attention to the successive conceptual upheavals attendant upon historical reconceptualizations that characterized broadened understandings especially in mathematics and the sciences (Davydov 1990). And algebraic structure constituted such a case. Bourbaki (1962; cited in Davydov 1975, p. 65) noted that the historical development of algebraic structure occurred in parallel with the development of the real numbers, accompanying each historical broadening of the concept of number, until algebraic structure exceeded the bounds of number, and the real number system became merely a special case of algebraic structure. What was required then, in order to render algebraic structure accessible to children, was the fusion of the highest level of generality with the culturally and historically necessary initial and intermediary steps to the building of the real number system.

The fact that the algebraic structure of positive scalar quantities is shared by the real numbers became the key to maintaining a theoretical focus while at the same time allowing for the accommodation of children's learning needs. Children could study scalar quantities such as the length, area, volume, and weight of real objects, which they can access visually and tactilely, discern their properties, and in this way equivalently access the mathematical structure of the real number system. This is the approach taken by Davydov's elementary curriculum (Davydov, Gorbov, Mikulina, & Saveleva 1999), which stands as a major departure from conventional programs.

Such a curriculum is, moreover, clearly in violation of the stages of development proposed by Piaget. Vygotsky, however, did not accept Piaget's separation of instruction and development, nor the assumption that the latter had to precede the former. Rather, Vygotsky contended that *learning*, and in particular, the mastery of scientific concepts, *leads development*. He wrote, "The formal discipline of scientific concepts gradually transforms the structure of the child's spontaneous concepts and helps organize them into a system: this furthers the child's ascent to higher developmental levels". (Vygotsky 1986, p.206)

2. Role of Generalization and Abstraction in Initial Development

An orientation to algebraic structure requires a focus on the most general and abstract characteristics of real phenomena, beginning with children's initial classroom encounters with such phenomena. This, in turn, requires the development of voluntary attention. And as is the case with scientific concepts and theoretical learning in general, pedagogical mediation is necessary.

This is accomplished in Davydov's program by focusing children on the theoretical characteristics of real objects, objects with which they are familiar, asking them to compare such objects with respect to their length, area, volume, or weight, and to progressively refine such comparisons until they culminate in measurement itself. Small children typically make such comparisons when receiving along with a sibling or friend, a cookie, candy bar, or a glass of juice or cola. What parent is not familiar with the small accusatory finger pointing at the other's handout and asserting that, "his cookie is bigger" or she "has more juice"? Davydov's program takes children painstakingly through all the refinements of quantitative comparison, while at the same time confronting them with tasks in which they determine that when adding a volume A to a volume B, for example, the result is the same as when the order of addition is reversed. Repeating this task in combining two different lengths of wood, or adding the weight of a pinecone and a pattern block in either order, children are developing the commutative property of addition of positive scalar quantities. Since they do not know the actual length of the wood or weight of the objects (that would require measurement, which will come later), the children label their result with letters, such as T + C = C + T, with the understanding that such a result is generalizable to any two quantities. It is noteworthy that initially the mere prizing out of the quantity from the empirical features, such as the shape and color, of the objects in question, is the beginning of a theoretical orientation to the task.

Once children's comparisons have progressed to the actual measurement of quantities (through the laying off of a part of the quantity arbitrarily designated as a unit), a number is generated that is the measure of the quantity by the unit employed. The properties of quantities apply also to their numerical designations, which vary with the unit of measurement. Children have already established these generalized properties for *any* quantities, hence their extension to specific numerical designations of quantities is simply a concrete application of a general result previously obtained and expressed in the symbolism of algebra. Now algebra is no longer initially learned as a generalization of arithmetic, but rather as a generalization of the relationships between quantities and the properties of actions on quantities. With the introduction of measurement and the definition of number emanating there from, the application of the properties of quantities to their numerical designations represents the ascent from the abstract to the concrete. This ascent "fills in" as it were, the conceptual content, as the abstraction is recognized in and enriched by its myriad concrete embodiments. (cf. Schmittau & Morris 2004.)

3. Ascent from the Abstract to the Concrete

By developing number from the measurement of quantities, Davydov's curriculum also breaks with the common practice of beginning formal mathematical study with number. Observing that culturally and in individual development, the concept of quantity is prior to that of number, he indicted the rush to number as a manifestation of ignorance of the real origins of concepts (Davydov 1990). In his first grade curriculum (Davydov et al. 1999), so extensive is the foundation of investigation of (mostly continuous) quantities that number does not appear until the second semester.

During the second semester of the first grade there is a major emphasis on the concept of positionality and the study of various number system bases. This, too, is for the purpose of establishing a sufficiently generalized conceptual base for operations with whole numbers, and has foundational applicability to decimals and polynomials as well. Vygotsky addressed this need as follows:

> "As long as the child operates with the decimal system without having become conscious of it as such, he has not mastered the system, but is, on the contrary, bound by it. When he becomes able to view it as a particular instance of the wider concept of a scale of notation, he can operate deliberately with this or any other numerical system. The ability to shift at will from one numerical system to another (e.g., to "translate" from the decimal system into one that is based on five) is the criterion of this new level of consciousness, since it indicates the existence of a general concept of a system of numeration". (Vygotsky 1986, p. 203)

Emphasis in Davydov's first grade curriculum is on the development of such general concepts. When we implemented Davydov's elementary mathematics program in a school setting in the US, a few parents expressed concerns that their children might be "lagging behind" other first graders who were learning to add and subtract, while our children were busy measuring and developing concepts of continuous quantity and positionality. However, six weeks into the first semester of second grade, the children studying Davydov's curriculum were accurately and with understanding, subtracting four digit numbers with three regroupings (Schmittau 2003b), easily surpassing their counterparts in both conventional and "reform" mathematics programs, and demonstrating that theoretical understanding and computational competence are not mutually exclusive, but rather complementary.

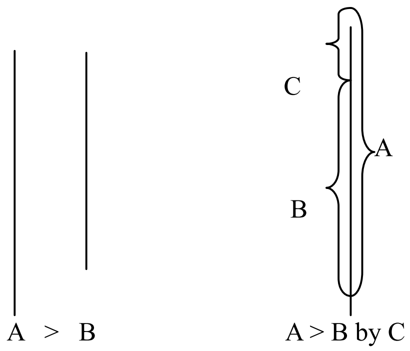
Nor was this sort of phenomenon limited to the addition and subtraction of whole numbers. A great deal of time was typically invested in developing a broad and deep conceptual foundation, after which progress was, by US standards, uncommonly rapid. By the end of the third grade curriculum, the children studying Davydov's program were solving applied problems involving proportional reasoning, differential rates of work, and rate, time and distance, that continued to challenge eleventh graders in their second year of formal algebra study in our regional high schools. Moreover, although Davydov's third grade curriculum (the third year is the final year of Russian elementary school) consisted of 969 problems (each year of the curriculum was composed entirely of very deliberately designed and sequenced problems), the class's teacher reported that students had no difficulties with the last 400 problems, despite the fact that objectively the level of difficulty of the problems themselves had increased considerably as indicated above. In fact, the more complex and difficult the problems became, the less difficulty the children had in solving them, and the more rapidly they progressed through them.

This attests to the value of laying a broad in-depth conceptual foundation with a focus on theoretical (i.e., algebraic) structure, although this initially appears counterintuitive to US teachers and administrators who have become conditioned to the belief that "covering" the same topics year after year is the only way to deal with students' failure to master them.

4. Appropriation of Psychological Tools

There is, however, another characteristic of the Vygotskian approach to the development of algebraic thinking that is also of critical importance for student mastery. It is the representational "schematic". By the time children reach the second grade, they generally have little need of actual objects, preferring to work instead with the "schematics' that focus them on, isolate, and express the mathematical actions in which they are engaged. These schematics are representations, but are not pictorial in nature as is generally the case when US children model problem situations.

Early in the first grade, for example, children determine that they can make two unequal volumes equal by adding to the smaller or subtracting from the larger the difference between the original quantities. They determine that if volume A is greater than volume B, A = B + C, where C is the difference between A and B. The children may schematize their result with a "length" model as in Fig. 1, and symbolize it with equations.



Now A = B + C
B = A - C
C = A - B

Figure 1: Schematic representing the change necessary to equalize two volumes, A and B.

The following problem, which is representative of problems occurring approximately half way through the first grade curriculum, provides another example of the role of the schematic in problem solving: N apples were in a bowl on the table. R people entered the room and each took an apple. How many apples remained? Children first analyze the structure of the problem, identifying it as a part-whole structure, with N as the whole and R as a part. They schematize the quantitative relations expressed in the problem as follows:

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  N
 / \
R   ?
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Since they know the whole and are trying to find a part, they know the missing part must be the difference between the whole and the known part, i.e., N – R. Later when they are confronted with a set of problems and told to solve those that are solvable and to change any that are not in order to make them solvable, they will not write "16" or "2" as solutions for a problem such as the following:

There were 7 books on the bookshelf. Nine children entered the library and each child took a book from the shelf. How many books are now on the shelf?

Instead they will recognize that a part cannot be larger than the whole, will identify the problem as a "trap" (i.e., unsolvable), and change it along the lines of the following examples.

A) There were seven books on the shelf. Five children entered the library and each removed a book from the shelf. How many books are now on the shelf?

B) There were 27 books on the shelf. Nine children entered the library and each took a book from the shelf. How many books are now on the shelf?

C) There were seven books on the shelf. Nine children entered the library and each placed a book on the shelf. How many books are now on the shelf?

Children recognize that either the whole must be increased (B) or the part decreased (A) (occasionally they may do both), or the *action* on the objects must be changed. In (C), for example, both the books *initially* on the shelf and those *added* by the children, are now *parts*, whereas in the other formulations the first was the whole and the second was a part.

One may discern in these three problems an ascent from the abstract to the concrete. From the three equations generated from the problem arising from the comparison of two volumes (Fig. 1), children discern their application to the part-whole situation in the problem of discrete quantity (apples), and to the specific numerical designations of quantities in the book problem, where they must reason in reverse, as it were, that the whole must be greater than the parts in order for the problem to have a solution. Because they have a theoretical orientation to the problem structure, they can analyze the relationships between quantities (of apples) without any numerical designations to provide them with cues. And in the book problem they are not led by the numerical aspects of the problem to simply add the two numbers or subtract the smaller from the larger.

5. Some Observations from Research with Russian and U.S. Students

I first observed the results of this approach centering on algebraic structure and subsequently "ascending" to concrete numerical applications while in Russia observing classes of children who were being instructed with Davydov's elementary curriculum. There, beginning third graders solved the literal equation below as follows:

$$x \bullet a \div k = b - e$$
$$(x \bullet a) \div k = (b - e)$$
$$(x \bullet a) = k \bullet (b - e)$$
$$x = k \bullet (b - e) \div a$$

Then they solved the following problems: $615 \bullet 70 - x = 39218$ and $x + 643 \div 50 = 478$.

The equations involving numbers were concrete applications of their previous algebraic work solving literal equations, for which they had to rely on their knowledge of algebraic structure alone. The Russian children were also assigned tasks that reached a level of abstraction where even letters were eliminated. Beginning fourth graders, for example, were given the following problem of order of operations:

O • O + O • (O - O • O) ÷ O where terms and factors were represented only by circles.

In addition, in my research into the mathematical understandings of the Russian children, I found them consistently reasoning from abstract and generalized (algebraic) understandings rather than the properties of numbers. When, for example, a beginning fifth grade girl was asked to consider the meaning of the monomial product "a•b", she asserted that she wanted to reconsider previous instances of multiplication about which she had been questioned. Then she proceeded to group these instances—of multiplication of positive integers, fractions, a positive times a negative, and two negatives—under the designation "a•b", asserting that it was a general expression under which all of these special cases could be subsumed. Her response gave evidence of having, through mastery of algebraic concepts, "gained a vantage point from which [she saw] concepts of arithmetic in a broader perspective" (Vygotsky 1986, p. 202). And it was in marked contrast to the responses of US high school and university students who also participated in the study, and who overwhelmingly substituted small whole numbers for "a" and "b", and then described the meaning of this generalized product as, for example, five groups of three objects. It was rare that a US student expressed any awareness that "a" and "b" could be any real numbers, and that restricting these variables to small whole numbers was a distortion of their meaning. Despite the fact that the older US students had had formal courses in algebra, their responses consistently attested to the fact that they were not operating from such a broadened perspective. The "new, higher concepts" had not, in their case, "transformed the meaning of the lower" (Vygotsky 1986, p.202), but instead had been reduced to the meaning of the lower, as they apparently attempted to subsume these algebraic generalizations into a schema predicated on counting number. (See Schmittau 2003a, for a more extensive discussion).

It is important to note that the "new, higher concepts" cannot "transform the meaning of the lower" if the conceptual schema does not *subsume* these "new, higher concepts". Therefore, Hegel's admonition not to allow the child to continue for too long in an empirical mode of thinking, and Davydov's early introduction of algebraic structure as foundational, reflect a critically important principle, and one of which modern psychology is well aware. The formation of the initial schema on a basis of sufficient generality is of utmost importance; since the restructuring of an entrenched and inadequate schema is a daunting cognitive undertaking of very uncertain outcome. Skemp (1987) has written extensively on the cognitive difficulty such restructurings of an established schema require, and Davydov (1990) and Kline (1959) have noted the upheavals such collective restructurings produced in the history of mathematics.

In an investigation of Davydov's elementary curriculum in what is, to the best of our knowledge, its first implementation in the US, we found that our US children evinced generalized understandings not unlike those of their Russian counterparts. They were aided in developing their knowledge of algebraic structure by the schematic models, which functioned as "helpers" in the children's language and as psychological tools within the framework of Vygotskian theory (Vygotsky 1986).

Early in the second grade curriculum, for example, the US children encountered the problem: What is T – 4 – 4? After much discussion, the debate centered on two possible answers: T- 0 (= T) or T –8. Neither side in the discussion could convince the other of their point of view. Finally, as the debate entered its second day, the teacher asked a child who argued that the answer was T – 0 to mark the parts and the whole in the statement: 14 – 4 – 4 = 14 (since the child had argued that T – 4 – 4 = T). The child marked the 14 as the whole in both places where it appeared in the statement of equality, and marked each 4 as a part. The teacher then asked the child to mark the whole and the parts in the statement: P – A – B = C. Here the child correctly marked P as the whole, and A, B, and C as the parts, and then drew the following schematic:

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   P
 / | \
A  B  C
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When the child realized that a similar schematic of the earlier answer would have to show 14 as the whole and 4, 4, and 14, as the parts, the child realized that in the original answer 14 had been treated simultaneously as the whole and also as one of the parts, and at the suggestion of classmates, changed the parts to 4, 4, and x (Lee 2002). What is striking here is that the child did not see the error until the relationship between the quantities was expressed algebraically in the abstract. The numbers in the problem had been the source of the error; the numbers had misled the child and the abstract algebraic representation of the mathematical structure corrected the error.

A negative illustration of the role of such psychological tools occurred in the case of a US child who was very quick with numerical computation, and refused to use schematics once numbers were introduced, protesting that they were unnecessary and cumbersome. This child was able to solve one-step word problems fairly well, but when two-step problems were introduced, the child "just picked numbers and calculated" (Lee 2002), ignoring the internal relationships among the quantities, and making many errors. And when no numbers were present, as in the following problem, the resulting reduction to "A + B" was predictable: Tanya picked A mushrooms. Her grandma picked B mushrooms more than Tanya did. How many mushrooms did grandma pick? (Davydov, Gorbov, Mikulina, & Saveleva 2000, p. 54). The child was even less able to solve the multi-step problems in the third grade curriculum, the result of failure to master the requisite tools of analysis. What is significant is that this child showed considerable ability to reason mathematically in the first grade, but as a result of failure to master the important semiotic tools of analysis, this capability did not develop along the lines of its initial promise.

According to Vygotsky, psychological tools, unlike their material counterparts that are directed toward action on the outer environment, are directed inward toward the control of one's own behavior. What is the direction that the schematic gives to the problem solver? It orients the child's attention to the internal relations obtained among quantities, rather than to the empirical features of the problem or familiar aspects of numbers.

The emphasis on theoretical analysis of algebraic structure in Davydov's program resulted in high levels of problem solving and accurate computation as well. By the completion of the third and final year of Davydov's elementary curriculum, the US children accurately solved computational problems requiring the multiplication of four–digit numbers and the division of six-digit numbers by three-digit numbers. They had no need of calculators and did not use them. Their theoretical understandings enabled these concrete numerical applications (Schmittau 2004). The assumption that the development of mathematical understanding precludes the development of computational facility, is an unfortunate accompaniment of the current "reform" movement in the US.

6. Summary

It is interesting that in 1963, Zankov proposed a reform of school mathematics in Russsia which shares some commonality with the current US reform, in which he advocated an "approach to structuring the learning process in which the emphasis shifts to the pupil's independent intellectual activity". (Elkonin 1975, p.37) Although noting that his proposal had been implemented in several schools and some improvement had been reported, Elknonin rejected Zankov's notion, citing Vygotsky's position that the *content* of instruction was more important than the method, and arguing that one could not fix the weaknesses of a curriculum with a change in teaching methodology. A theoretical approach to mathematics was essential.

> "What is original with Vygotskii is not his general view of the role of instruction in development, but that he saw the source of this role in the *content* of the knowledge being acquired, in the mastery not of empirical concepts but of scientific ones, which calls for a special form of instruction". (Elkonin 1975, p. 37)

Such a special form of instruction has been developed for the teaching of elementary mathematics, in the form of Davydov's three-year elementary mathematics curriculum that focuses on algebraic structure and then applies algebraic understandings to concrete numerical instances. This organization of learning activity for the development of algebraic thinking is very different from the introduction of elements of algebra into the study of arithmetic in the early grades. The intended theoretical (algebraic) understanding is attained through appropriation of psychological tools in the form of specially designed schematics, whose mastery is not merely incidental to but the explicit focus of instruction.

It is also a matter of more than passing significance that this approach to the development of algebraic thinking, as is typical of research grounded in Vygotskian theory, had its origins in the profound study of the cultural historical development of mathematics rather than investigations of individual children's mathematical thinking. Here is another manifestation of significant divergence between Piagetian and Vygotskian theory. Vygotsky held that psychological phenomena occurred first on the social and secondarily on the individual plane.

Nicolopoulou (1997) asserts that our psychological studies must be historically grounded. "If we limit ourselves to studying individual development, we can never convincingly capture the ways in which the endpoints of individual development are not universal but culturally shaped, a fact that historical studies bring immediately into proper focus". (Nicolopoulou 1997, p. 223)

It is also important to remember that the characteristics of the Vygotskian approach to the development of algebraic thinking—cultural historical grounding, initial development from the general structural foundation, ascent from the abstract to the concrete, and appropriation of psychological tools—are all interrelated and function synergistically. Selective or piecemeal implementation of them could scarcely be expected to yield equivalent results.

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